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Tension of a long circular cylinder having a spherical cavity with a peripheral edge crack

Doo-Sung Lee *

Department of Mathematical Sciences, University of Delaware, Newark, DE 19711, USA

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Abstract

The singular stress problem of a peripheral edge crack around a spherical cavity in a long circular cylinder under tension is investigated. The problem is solved by using integral transforms and is reduced to the solution of three integral equations. The solution of these equations is obtained numerically by the method due to Erdogan, Gupta, and Cook, and the stress intensity factors are displayed graphically.

Keywords: Tension; Cylinder; Spherical cavity; Peripheral edge crack; SIF

1. Introduction

It has been known for a long time that the presence of pores in a brittle solid seriously degrades its strength. Strength reduction from porosity was firstly viewed as resulting stress concentration effects magnifying the stress intensity factor on nearby flaws. Fractures emanating from spherical cavities are of practical importance in the design of various structures.

The problem of a crack emanating from a sphere or mixed boundary value problems for regions with spherical boundary have been investigated by various researchers. Srivastav and Narain (1965) investigated the solution of Laplace's equation in a sphere where the mixed type of conditions are specified on a diametral plane of a sphere. Their method was adopted by Srivastava and Dwivedi (1971) for the solution of penny shaped crack problem in a sphere. By the same method Dhaliwal et al. (1979) investigated the problem of penny shaped crack in a sphere embedded in an infinite medium.

On the other hand, Atsumi and Shindo (1983a,b) solved the problem of internal edge crack in spherical shell or around a spherical cavity by the application of the technique of Keer et al. (1976, 1977).

* Present address: College of Education, Konkuk University, 1 Hwayang-Dong, Kwangjin-Gu, Seoul, South Korea. Fax: +82-02-454-3864.

E-mail address: dslee@kkucc.konkuk.ac.kr (D.-S. Lee).

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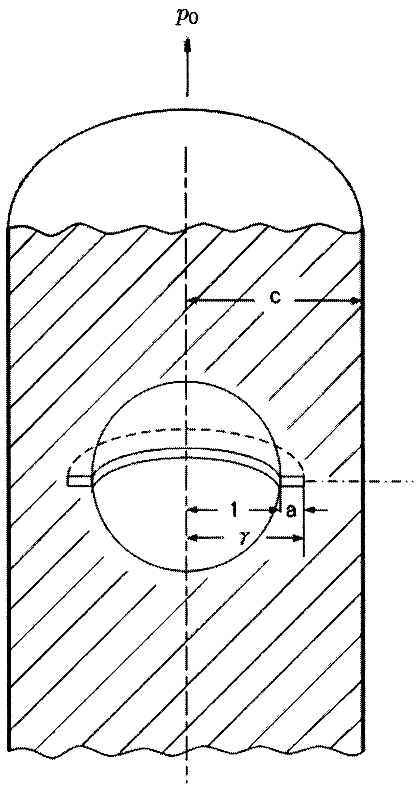

Fig. 1. Geometry.

However, the problem for the multiply connected region considered here does not appear to be investigated. In this paper, we consider the tension of a long circular cylinder having a spherical cavity with a peripheral edge crack as depicted in Fig. 1.

This solution is of practical applications since many mechanical parts are in the form of a cylinder which could contain pores and flaws in the process of manufacturing. The related problems of edge crack have been recently investigated by Lee (2002a,b).

We take the axis of the shaft as the z-axis and use polar coordinates ρ and ϕ for defining the position of an element in the place of a cross section.

We also use the spherical coordinates (r, ϕ, θ) which are connected with the cylindrical coordinates by

$$z = r\cos\theta, \quad \rho = r\sin\theta.$$

2. Formulation of the problem

Now, consider an infinite circular cylinder of radius c having a spherical cavity of radius unity, which is under tension by uniform axial forces as shown in Fig. 1.

For convenience, the center of the spherical cavity will be taken as the origin.

The geometry of Fig. 1 is applicable and the boundary conditions are

$$u_z(\rho, 0) = 0, \quad \gamma \leqslant r < c, \tag{2.1}$$

$$\sigma_{zz}(\rho,0)=0,\quad 1\leqslant r<\gamma, \tag{2.2}$$

$$\sigma_{\rho z}(\rho,0)=0,\quad 1\leqslant \rho\leqslant c, \tag{2.3}$$

$$\sigma_{\rho z}(c,z)=0, \tag{2.4}$$

$$\sigma_{\rho\rho}(c,z)=0, \tag{2.5}$$

$$\sigma_{rr}(1,\theta)=0, \tag{2.6}$$

$$\sigma_{r\theta}(1,\theta)=0. \tag{2.7}$$

Moreover,

$$\sigma_{zz}=p_0,\quad z=\pm\infty. \tag{2.8}$$

A needed representation of the displacement D for the present problem can be obtained from Collins (1962), and it is

$$D=D_1+D_2. \tag{2.9}$$

The representation D_1 is

$$D_1=(3-4v)\psi\mathbf{e}_z-z\,\mathrm{grad}\,\psi-\mathrm{grad}\,\phi, \tag{2.10}$$

where $\mathbf{e}_z$ is the unit vector in the z-direction, v is Poisson's ratio, and $\psi(\rho,z)$ and $\phi(\rho,z)$ are harmonic functions, and D_2 is deduced from the Papkovich–Neuber solution of the equations of elasticity. We have as a representation for D_2

$$D_2=(3-4v)\chi\mathbf{e}_\rho-\rho\,\mathrm{grad}\,\chi, \tag{2.11}$$

where $\mathbf{e}_\rho$ is the unit vector in the ρ-direction, and $\chi(\rho,z)\cos\phi$ is a harmonic function, that is $\chi(\rho,z)$ is a solution of the equation

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}-\frac{1}{\rho^2}\right)\chi=0. \tag{2.12}$$

The stresses corresponding to (2.9) are

$$\frac{\sigma_{\rho z}}{2\mu}=(1-2v)\frac{\partial\psi}{\partial\rho}-z\,\frac{\partial^2\psi}{\partial\rho\,\partial z}-\frac{\partial^2\phi}{\partial\rho\,\partial z}+(1-2v)\frac{\partial\chi}{\partial z}-\rho\,\frac{\partial^2\chi}{\partial\rho\,\partial z}, \tag{2.13}$$

$$\frac{\sigma_{zz}}{2\mu}=2(1-v)\frac{\partial\psi}{\partial z}-z\,\frac{\partial^2\psi}{\partial z^2}-\frac{\partial^2\phi}{\partial z^2}+\frac{2v}{\rho}\frac{\partial}{\partial\rho}(\rho\chi)-\rho\,\frac{\partial^2\chi}{\partial z^2}, \tag{2.14}$$

where μ is the modulus of rigidity.

Suitable representations of the functions $\phi(\rho,z),\psi(\rho,z)$, and $\chi(\rho,z)$ are

$$\phi(\rho,z)=(2v-1)\int_0^\infty\xi^{-1}A(\xi)J_0(\xi\rho)\mathrm{e}^{-\xi z}\,\mathrm{d}\xi+\int_0^\infty\xi^{-2}B(\xi)I_0(\xi\rho)\cos\xi z\,\mathrm{d}\xi$$
$$+\sum_{n=0}^{\infty}a_nr^{-(2n+1)}P_{2n}(\cos\theta)+\frac{p_0v}{2(1+v)}(\rho^2-2z^2), \tag{2.15}$$

$$\psi(\rho,z)=\int_0^\infty A(\xi)J_0(\xi\rho)\mathrm{e}^{-\xi z}\,\mathrm{d}\xi+\sum_{n=0}^{\infty}b_nr^{-(2n+2)}P_{2n+1}(\cos\theta)+\frac{p_0}{2(1+v)}z, \tag{2.16}$$

$$\chi(\rho,z) = \int_0^\infty \xi^{-1} C(\xi) I_1(\xi\rho) \cos \xi z \, d\xi, \tag{2.17}$$

where J_0 is the Bessel function of the first kind and I_0 and I_1 are the modified Bessel functions of the first kind and $P_n(x)$ is the Legendre polynomial respectively, and $A(\xi), B(\xi)$, $C(\xi), a_n$ and b_n are unknown functions and unknown coefficients to be determined later. With these choice of functions, we can immediately see that conditions (2.3) and (2.8) are automatically satisfied.

If we substitute functions (2.15)–(2.17) into the displacement field (2.9), and expression (2.14) for σ_{zz}, the expressions for σ_{zz} and u_z on the plane $z = 0$ are given by the equations

$$u_z = -\alpha \int_0^\infty A(\xi) J_0(\xi r) \, d\xi, \tag{2.18}$$

$$\begin{aligned} \sigma_{zz} = 2\mu \Bigg[& -\int_0^\infty \xi A(\xi) J_0(\xi r) \, d\xi + \sum_{n=0}^\infty r^{-(2n+3)} P_{2n}(0)(2n+1)\{(2n+1)a_n + \alpha b_n\} \\ & + \int_0^\infty [B(\xi) I_0(\xi\rho) + C(\xi)\{I_0(\xi\rho)2v + \xi\rho I_1(\xi\rho)\}] \, d\xi + p_0 \Bigg], \end{aligned} \tag{2.19}$$

where $\alpha = 2(1 - v)$.

To satisfy the boundary condition (2.1), the following definition is made on the crack opening displacement

$$\begin{aligned} \frac{1}{\alpha} \frac{\partial u_z(\rho,0)}{\partial \rho} &= -g(\rho), \quad 1 \leqslant \rho \leqslant \gamma, \\ &= 0, \quad \gamma \geqslant \rho. \end{aligned} \tag{2.20}$$

From (2.20), $A(\xi)$ is determined as

$$A(\xi) = \int_1^\gamma t g(t) J_1(\xi t) \, dt, \tag{2.21}$$

which satisfies (2.2) automatically, and if we substitute this form into (2.19), and use (2.2) it reduces that equation to

$$\begin{aligned} & -\frac{2}{\pi} \int_1^\gamma t g(t) R(r,t) \, dt + \sum_{n=0}^\infty r^{-(2n+3)} P_{2n}(0)(2n+1)\{(2n+1)a_n + \alpha b_n\} \\ & + \int_0^\infty [B(\xi) I_0(\xi\rho) + C(\xi)\{I_0(\xi\rho)2v + \xi\rho I_1(\xi\rho)\}] \, d\xi = -p_0, \quad 1 \leqslant \rho \leqslant \gamma, \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} R(r,t) &= \frac{1}{r^2 - t^2} E\left(\frac{r}{t}\right), \quad t > r, \\ &= \frac{r}{t} \frac{1}{r^2 - t^2} E\left(\frac{t}{r}\right) - \frac{1}{rt} K\left(\frac{t}{r}\right), \quad r > t. \end{aligned} \tag{2.23}$$

K and E in (2.23) are complete elliptic integrals of the first and the second kind, respectively.

The solution will be complete, if the conditions on the surface of the cylinder and spherical cavity are satisfied.

3. Conditions on the surface of the cylinder

Eq. (2.22) gives one relation connecting the unknown functions $g(t), B(\xi), C(\xi)$ and unknown coefficients a_n and b_n. Two among the remaining relations are given by the conditions on the curved surface $\rho = c$. The stress component besides (2.13) and (2.14) which is needed for the present analysis is given by the following equation

$$\frac{\sigma_{\rho\rho}}{2\mu} = 2v\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial\rho^2} - \frac{\partial^2\phi}{\partial\rho^2} + 2(1-v)\frac{\partial\chi}{\partial\rho} + 2v\frac{\chi}{\rho} - \rho\frac{\partial^2\chi}{\partial\rho^2}. \tag{3.1}$$

An expression useful for the present analysis is the following

$$\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right) = (-1)^n n!\frac{P_n(\cos\theta)}{r^{n+1}}. \tag{3.2}$$

It is easily shown that by employing formula (3.2) the value of $\sigma_{\rho z}$ on the surface $\rho = c$ is given by

$$\begin{aligned}\frac{\sigma_{\rho z}(c,z)}{2\mu} &= \int_0^\infty [B(\xi)I_1(\xi c) + C(\xi)\{\xi c I_0(\xi c) - 2(1-v)I_1(\xi c)\}]\sin\xi z\,\mathrm{d}\xi - z\int_0^\infty A(\xi)\xi^2 J_1(\xi c)\mathrm{e}^{-\xi z}\,\mathrm{d}\xi \\ &\quad + \sum_{n=0}^\infty a_n\frac{1}{(2n)!}\frac{\partial^{2n+1}}{\partial z^{2n+1}}\frac{c}{r_c^3} - \sum_{n=0}^\infty b_n\frac{1}{(2n+1)!}\left(z\frac{\partial}{\partial z} - 1 + 2v\right)\frac{\partial^{2n+1}}{\partial z^{2n+1}}\frac{c}{r_c^3},\end{aligned} \tag{3.3}$$

where $r_c = \sqrt{c^2+z^2}$.

Boundary conditions (2.4) and (2.5) can be written in the alternative forms

$$\mathscr{F}_s[\sigma_{\rho z}(c,z); z \to \xi] = 0, \tag{3.4}$$

$$\mathscr{F}_c[\sigma_{\rho\rho}(c,z); z \to \xi] = 0. \tag{3.5}$$

If we substitute (3.3) into (3.4), we obtain the equation

$$\begin{aligned}&B(\xi)I_1(\xi c) + C(\xi)\{\xi c I_0(\xi c) - 2(1-v)I_1(\xi c)\} \\ &\quad = \frac{2}{\pi}\left[\int_0^\infty 2A(\zeta)\frac{J_1(\zeta c)\zeta^3\xi}{(\zeta^2+\xi^2)^2}\,\mathrm{d}\zeta + \sum_{n=0}^\infty a_n\frac{(-1)^n}{(2n)!}\xi^{2n+2}K_1(\xi c)\right. \\ &\qquad \left. - \sum_{n=0}^\infty b_n\frac{(-1)^{n+1}}{(2n+1)!}\xi^{2n+2}\{(2n+3-2v)K_1(\xi c) - \xi c K_0(\xi c)\}\right],\end{aligned} \tag{3.6}$$

where integration by parts and the following formula are used

$$\int_0^\infty \frac{c}{(c^2+z^2)^{3/2}} = \xi K_1(\xi c). \tag{3.7}$$

Similarly, the value of $\sigma_{\rho\rho}$ on the surface $\rho = c$ is given by

$$\begin{aligned}\frac{\sigma_{\rho\rho}(c,z)}{2\mu} &= \int_0^\infty \left[B(\xi)\left\{\frac{I_1(\xi c)}{\xi c} - I_0(\xi c)\right\} + C(\xi)\left\{(3-2v)I_0(\xi c) - \xi c I_1(\xi c) - \frac{4(1-v)}{\xi c}I_1(\xi c)\right\}\right]\cos\xi z\,\mathrm{d}\xi \\ &\quad - \int_0^\infty A(\xi)\xi\left[J_0(\xi c) - \frac{1-2v}{\xi c}J_1(\xi c) + \xi z\left\{J_0(\xi c) - \frac{1}{\xi c}J_1(\xi c)\right\}\right]\mathrm{e}^{-\xi z}\,\mathrm{d}\xi \\ &\quad - \sum_{n=0}^\infty a_n\frac{1}{(2n)!}\frac{\partial^{2n+2}}{\partial c^2\,\partial z^{2n}}\frac{1}{r_c} + \sum_{n=0}^\infty b_n\frac{1}{(2n+1)!}\left(z\frac{\partial^2}{\partial c^2} + 2v\frac{\partial}{\partial z}\right)\frac{\partial^{2n+1}}{\partial z^{2n+1}}\frac{1}{r_c}.\end{aligned} \tag{3.8}$$

If we substitute from (3.8) into (3.5), we obtain another equation

$$B(\xi)\left\{\frac{I_1(\xi c)}{\xi c}-I_0(\xi c)\right\}+C(\xi)\left\{(3-2v)I_0(\xi c)-\xi cI_1(\xi c)-\frac{4(1-v)}{\xi c}I_1(\xi c)\right\}$$
$$=\frac{2}{\pi}\left[\int_0^\infty 2A(\zeta)\zeta^2\left\{\frac{J_0(\zeta c)\xi^2}{(\zeta^2+\xi^2)^2}+\frac{J_1(\zeta c)(v(\zeta^2+\xi^2)-\xi^2)}{\zeta c(\zeta^2+\xi^2)^2}\right\}\mathrm{d}\zeta+\sum_{n=0}^\infty a_n\frac{(-1)^n}{(2n)!}\xi^{2n+2}\left\{K_0(\xi c)+\frac{K_1(\xi c)}{\xi c}\right\}\right.$$
$$\left.+\sum_{n=0}^\infty b_n\frac{(-1)^n}{(2n+1)!}\xi^{2n+2}\left\{2(n+1-v)K_0(\xi c)+\left(\frac{2n+1}{\xi c}-\xi c\right)K_1(\xi c)\right\}\right]. \tag{3.9}$$

If we put the value of $A(\zeta)$ given by (2.21) into the first term of the right-hand side of (3.6), it is evaluated as

$$\int_0^\infty 2A(\zeta)\frac{J_1(\zeta c)\zeta^3\xi}{(\zeta^2+\xi^2)^2}\,\mathrm{d}\zeta=2\int_1^\gamma tg(t)\,\mathrm{d}t\xi\left(1+\frac{\xi}{2}\frac{\partial}{\partial\xi}\right)\frac{\partial}{\partial c}i(\xi,c)$$
$$=-\xi\int_1^\gamma tg(t)\{\xi tI_0(\xi t)K_1(\xi c)-\xi cI_1(\xi t)K_0(\xi c)\}\,\mathrm{d}t, \tag{3.10}$$

where

$$i(\xi,c)=\int_0^\infty\frac{J_0(\zeta c)J_1(\zeta t)}{\zeta^2+\xi^2}\,\mathrm{d}\zeta=\xi^{-1}I_1(\xi t)K_0(\xi c),\quad c>t. \tag{3.11}$$

Similarly, the first term on the right-hand side of (3.9) is

$$\int_0^\infty 2A(\zeta)\zeta^2\left[\frac{J_0(\zeta c)\xi^2}{(\zeta^2+\xi^2)^2}+\frac{J_1(\zeta c)}{\zeta c}\left\{\frac{v}{\zeta^2+\xi^2}-\frac{\xi^2}{(\zeta^2+\xi^2)^2}\right\}\right]\mathrm{d}\zeta$$
$$=-\int_1^\gamma tg(t)\,\mathrm{d}t\left\{2\xi^2\left(1+\frac{\xi}{2}\frac{\partial}{\partial\xi}\right)-\frac{2}{c}\left(v+\frac{\xi}{2}\frac{\partial}{\partial\xi}\right)\frac{\partial}{\partial c}\right\}i(\xi,c)$$
$$=-\xi\int_1^\gamma tg(t)\left[\xi tI_0(\xi t)\left\{K_0(\xi c)+\frac{K_1(\xi c)}{\xi c}\right\}-I_1(\xi t)K_0(\xi c)\right.$$
$$\left.-\left\{\xi c+\frac{2(1-v)}{\xi c}\right\}I_1(\xi t)K_1(\xi c)\right]\mathrm{d}t. \tag{3.12}$$

4. Conditions on the surface of the spherical cavity

To satisfy the boundary conditions on the spherical cavity, it is necessary to express stresses in spherical coordinate system. They are as follows:

$$\frac{\sigma_{rr}}{2\mu}=-\frac{\partial^2\phi}{\partial r^2}-r\cos\theta\frac{\partial^2\psi}{\partial r^2}+\alpha\cos\theta\frac{\partial\psi}{\partial r}+(\alpha-2)\sin\theta\frac{\partial\psi}{r\partial\theta}+\alpha\sin\theta\frac{\partial\chi}{\partial r}-r\sin\theta\frac{\partial^2\chi}{\partial r^2}$$
$$+\frac{2-\alpha}{r}\left(\cos\theta\frac{\partial\chi}{\partial\theta}+\frac{\chi}{\sin\theta}\right), \tag{4.1}$$

$$\frac{\sigma_{r\theta}}{2\mu}=-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\cos\theta\frac{\partial^2\psi}{\partial r\partial\theta}-(\alpha-1)\sin\theta\frac{\partial\psi}{\partial r}+\frac{\alpha\cos\theta}{r}\frac{\partial\psi}{\partial\theta}$$
$$+\alpha\sin\theta\frac{\partial\chi}{r\partial\theta}+(\alpha-1)\cos\theta\frac{\partial\chi}{\partial r}-\sin\theta\frac{\partial^2\chi}{\partial r\partial\theta}. \tag{4.2}$$

To express the stress functions in terms of spherical coordinates, the following formulae in Whittaker and Watson (1973, p. 392) are useful

$$J_0(\xi\rho)e^{-\xi z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\{-\xi(z + ix\cos u + iy\sin u)\}\,du, \tag{4.3}$$

$$\int_{-\pi}^{\pi} (z + ix\cos u + iy\sin u)^n\,du = 2\pi r^n P_n(\cos\theta), \tag{4.4}$$

where $x = \rho\cos\phi$, $y = \rho\sin\phi$. Now

$$\phi^{(1)}(r,\theta) = -(\alpha - 1)\int_0^\infty \xi^{-1}A(\xi)J_0(\xi\rho)e^{-\xi z}\,d\xi = (\alpha - 1)\int_1^\gamma tg(t)\,dt\int_0^\infty \xi^{-1}J_1(\xi t)J_0(\xi\rho)e^{-\xi z}\,d\xi. \tag{4.5}$$

Using (4.3), (4.4) and the formula in Erdélyi et al. (1954)

$$\int_0^\infty \xi^{-1}J_1(\xi t)e^{-\xi\beta}\,d\xi = \frac{1}{t}\left(\sqrt{\beta^2 + t^2} - \beta\right)$$

to the last integral in (4.5), it reduces to

$$\begin{aligned}\int_0^\infty \xi^{-1}J_1(\xi t)J_0(\xi\rho)e^{-\xi z}\,d\xi &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty \xi^{-1}e^{-\xi\beta}J_1(\xi t)\,d\xi\,du = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{t}\left(\sqrt{\beta^2 + t^2} - \beta\right)du \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=0}^\infty \frac{(-1)^n(-\frac{1}{2})_n}{n!}\left(\frac{\beta}{t}\right)^{2n} - \frac{\beta}{t}\right]du \\ &= \sum_{n=0}^\infty \frac{(-1)^n(-\frac{1}{2})_n}{n!}\left(\frac{r}{t}\right)^{2n}P_{2n}(\cos\theta) - \frac{r}{t}P_1(\cos\theta),\end{aligned} \tag{4.6}$$

where $\beta = z + ix\cos u + iy\sin u$.

Also, if we use

$$I_0(\xi\rho)\cos\xi z = \sum_{n=0}^\infty (-1)^n\frac{(\xi r)^{2n}}{(2n)!}P_{2n}(\cos\theta) \tag{4.7}$$

and (4.5) and (4.6), we finally obtain for $\phi(r,\theta)$

$$\phi(r,\theta) = \sum_{n=0}^\infty P_{2n}(\cos\theta)\left[-(\alpha - 1)\frac{T_n}{2n}r^{2n} + a_n r^{-2n-1} + \frac{U_n}{2n}r^{2n}\right] - (\alpha - 1)\int_1^\gamma g(t)\,dt\,rP_1(\cos\theta), \tag{4.8}$$

where for brevity, we used

$$T_n = (-1)^n\frac{\Gamma(n - \frac{1}{2})}{\sqrt{\pi}\Gamma(n)}\int_1^\gamma g(t)t^{-2n+1}\,dt \tag{4.9}$$

and

$$U_n = \frac{(-1)^n}{(2n-1)!}\int_0^\infty B(\xi)\xi^{2n}\,d\xi. \tag{4.10}$$

Similarly if we use the formula

$$I_0(\xi\rho)\sin\xi z = \sum_{n=0}^\infty (-1)^n\frac{(\xi r)^{2n+1}}{(2n+1)!}P_{2n+1}(\cos\theta), \tag{4.11}$$

we can easily show that

$$\psi(r,\theta) = -\int_1^{\gamma} g(t)\,\mathrm{d}t + \sum_{n=0}^{\infty} P_{2n+1}(\cos\theta)\left(-T_{n+1}r^{2n+1} + b_{n+1}r^{-2n-2}\right). \tag{4.12}$$

If we use the formula

$$I_1(\xi\rho)\cos\xi z = -\sin\theta\sum_{n=1}^{\infty}(-1)^n\frac{(\xi r)^{2n-1}}{(2n)!}P'_{2n-1}(\cos\theta), \tag{4.13}$$

which can be obtained by differentiating (4.7) with respect to ρ, and where prime denotes the differentiation with respect to the argument, we obtain

$$\chi(r,\theta) = \sin\theta\sum_{n=1}^{\infty}(-1)^n r^{2n-1}P'_{2n-1}(\cos\theta)S_n, \tag{4.14}$$

where, again we used for brevity

$$S_n = \frac{(-1)^n}{(2n)!}\int_0^{\infty} C(\xi)\xi^{2n-1}\,\mathrm{d}\xi. \tag{4.15}$$

Therefore if we put expression (4.8), (4.12) and (4.14) into (4.2), and use properties of Legendre functions, we obtain

$$\begin{aligned}\frac{\sigma_{r\theta}}{2\mu} = \sin\theta\sum_{n=1}^{\infty}P'_{2n}(\cos\theta)\bigg[&-a_n\frac{2n+2}{r^{2n+3}} + \frac{b_{n-1}}{r^{2n+1}}\frac{\alpha-4n^2}{4n-1} - \frac{b_{n+1}}{r^{2n+3}}\frac{(2n+2)(2n+1+2\alpha)}{4n+3}\\ &- T_n\frac{(2n-1)F(n)}{(4n-1)2n}r^{2n-2} - T_{n+1}\frac{F(n+1)}{4n+3}r^{2n} + U_n r^{2n-2} - S_n\frac{(2n-1)^2 2(\alpha-n)}{4n-1}r^{2n-2}\\ &+ S_{n+1}\frac{2n+2}{4n+3}F(n+1)r^{2n} - \frac{p_0}{3}\delta_{1,n}\bigg],\end{aligned} \tag{4.16}$$

where $\delta_{1,n}$ is the Kronecker delta and $F(n) = \alpha - (2n-1)^2$. Similarly, we obtain

$$\begin{aligned}\frac{\sigma_{rr}}{2\mu} = \sum_{n=1}^{\infty}P_{2n}(\cos\theta)\bigg[&-a_n\frac{(2n+1)(2n+2)}{r^{2n+3}} + \frac{b_{n-1}}{r^{2n+1}}\frac{2nG(n)}{4n-1}\\ &- \frac{b_{n+1}}{r^{2n+3}}\frac{(2n+1)(2n+2)(2n+1+2\alpha)}{4n+3} - T_n\frac{(2n-1)F(n)}{4n-1}r^{2n-2} - T_{n+1}\frac{2n+1}{4n+3}G(n-1)r^{2n}\\ &- U_n(2n-1)r^{2n-2} + S_n\frac{4n(2n-1)^2(\alpha-n-2)}{4n-1}r^{2n-2} + S_{n+1}\frac{(2n+1)(2n+2)}{4n+3}H(n)r^{2n} + \frac{2p_0}{3}\delta_{1,n}\bigg]\\ &+\frac{p_0}{3} - \frac{S_1}{3}2(3v+1) - \frac{2a_0}{r^3} - \frac{b_1(4\alpha+2)}{3r^3} - T_1\frac{\alpha-4}{3},\end{aligned} \tag{4.17}$$

where

$$G(n) = 2-\alpha-2n(2n+3), \quad H(n) = 2(n+1)(5-2\alpha+2n)-2+\alpha. \tag{4.18a, b}$$

Eqs. (4.16) and (4.17) are zero when $r = 1$, thus coefficients of Legendre polynomials are zero for each n. If we solve these coefficients simultaneously, we obtain

$$b_{n-1} = \frac{1}{I(n)}\left[(2n-1)\left\{T_{n+1}(2n+1)(4n-1) - T_nF(n)\frac{4n+1}{2n}\right\} - U_n4n(4n-1) + S_n2(2n-1)^2J(n) - S_{n+1}\frac{4n-1}{4n+3}(2n+1)(2n+2)K(n) + 5p_0\delta_{1,n}\right], \quad n \geqslant 1, \tag{4.19}$$

where

$$I(n) = 8n^2 - 4n + \alpha(4n+1), \tag{4.20}$$

$$J(n) = 4n^2 + 5n - \alpha(4n+1), \tag{4.21}$$

$$K(n) = 2(n+1)(4n+5-2\alpha) - 1 \tag{4.22}$$

and

$$-2a_0 - \frac{2}{3}b_1(2\alpha+1) - T_1\frac{\alpha-4}{3} + \frac{p_0}{3} - \frac{S_1}{3}2(3v+1) = 0. \tag{4.23}$$

If we substitute a_n and b_n into (2.22), (3.6) and (3.9), we finally obtain following three integral equations

$$\frac{2}{\pi}\int_1^\gamma tg(t)\{R(r,t)+S(r,t)\}\,dt + \int_0^\infty B(\xi)K_{12}(r,\xi)\,d\xi + \int_0^\infty C(\xi)K_{13}(r,\xi)\,d\xi = \left[1+\frac{p_1}{r^3}+\frac{p_2}{r^5}\right]p_0, \quad 1 \leqslant r < \gamma, \tag{4.24}$$

where

$$p_1 = \frac{5v-4}{2(5v-7)}, \quad p_2 = -\frac{9}{2(5v-7)},$$

with

$$S(r,t) = \frac{\pi}{2}\left[-\frac{1+v}{3}\frac{1}{r^3t^2} + \sum_{n=1}^{\infty}S_n(r,t)\right],$$

$S_n(r,t)$ is listed in Atsumi and Shindo (1983b).

$$\frac{2}{\pi}\int_1^\gamma tg(t)K_{n1}(t,\xi)\,dt + \int_0^\infty B(\eta)K_{n2}(\xi,\eta)\,d\eta + \int_0^\infty C(\eta)K_{n3}(\xi,\eta)\,d\eta + B(\xi)h_{n1}(\xi,c) + C(\xi)h_{n2}(\xi,c) = f_n(\xi), \quad (n=2,3), \tag{4.25}$$

where

$$f_2(\xi) = \frac{2}{\pi}p_0\xi^2\left\{\frac{(5v-9-\xi^2)K_1(\xi c) + 5\xi cK_0(\xi c)}{2(7-5v)}\right\},$$

$$f_3(\xi) = \frac{2}{\pi}p_0\xi^2\left[(5v-4)K_0(\xi c) - \xi^2\left\{K_0(\xi c) + \frac{K_1(\xi c)}{\xi c}\right\} + \left(5\xi c + \frac{1-5v}{\xi c}\right)K_1(\xi c)\right]\frac{1}{2(7-5v)}.$$

We exempt the detailed expressions of K_{ij} and h_{ij} which are available on request.

A quantity of physical interest is the stress intensity factor which is given as

$$K = \lim_{r\to\gamma^+}\sqrt{2(r-\gamma)}\,\sigma_{zz}(r,0). \tag{4.26}$$

We let

$$r = \frac{a}{2}(s+1) + 1, \quad t = \frac{a}{2}(\tau+1) + 1 \tag{4.27}$$

and in order to facilitate numerical analysis, assume $g(t)$ to have the following form:

$$g(t) = p_0(t-1)^{1/2}(\gamma - t)^{-1/2}G(t). \tag{4.28}$$

With the aid of (4.27), $g(\tau)$ can be rewritten as

$$g(\tau) = p_0 G(\tau)\left(\frac{1+\tau}{1-\tau}\right)^{1/2}. \tag{4.29}$$

The stress intensity factors K can therefore be expressed in terms of $G(t)$ as

$$K/p_0 = \sqrt{2a}G(\gamma) \tag{4.30}$$

or in terms of the quantitiy actually calculated

$$K/p_0\sqrt{a} = \sqrt{2}G(\gamma). \tag{4.31}$$

5. Numerical analysis

In order to obtain numerical solutions of (4.24) and (4.25), substitutions are made by the application of (4.27) and (4.29) to obtain equations of following forms:

$$\begin{aligned}\frac{a}{\pi}\int_{-1}^{1}\left(\frac{1+\tau}{1-\tau}\right)^{1/2} G(\tau)\left[\frac{a}{2}(\tau+1)+1\right][R(s,\tau)+S(s,\tau)]\,\mathrm{d}\tau + \int_0^\infty \widetilde{B}(\xi)K_{12}(s,\xi)\,\mathrm{d}\xi \\ + \int_0^\infty \widetilde{C}(\xi)K_{13}(s,\xi)\,\mathrm{d}\xi = 1 + \frac{p_1}{r^3} + \frac{p_2}{r^5}, \quad -1 < s < 1,\end{aligned} \tag{5.1}$$

$$\begin{aligned}\frac{a}{\pi}\int_{-1}^{1}\left(\frac{1+\tau}{1-\tau}\right)^{1/2} G(\tau)\left[\frac{a}{2}(\tau+1)+1\right]K_{n1}(\tau,\xi)\,\mathrm{d}\tau + \widetilde{B}(\xi)h_{n1}(\xi,c) + \widetilde{C}(\xi)h_{n2}(\xi,c) \\ + \int_0^\infty \widetilde{B}(\eta)K_{n2}(\xi,\eta)\,\mathrm{d}\eta + \int_0^\infty \widetilde{C}(\eta)K_{n3}(\xi,\eta)\,\mathrm{d}\eta = \tilde{f}_n(\xi), \quad (n=2,3)\ \ 0 \leqslant \xi < \infty,\end{aligned} \tag{5.2}$$

where $\widetilde{B}(\eta) = B(\eta)/p_0$, etc. The numerical solution technique is based on the collocation scheme for the solution of singular integral equations given by Erdogan et al. (1973). This amounts to applying a Gaussian quadrature formula for approximating the integral of a function $f(\tau)$ with weight function $[(1+\tau)/(1-\tau)]^{1/2}$ on the interval [−1,1]. Thus, letting n be the number of quadrature points,

$$\int_{-1}^{1}\left(\frac{1+\tau}{1-\tau}\right)^{1/2} f(\tau)\,\mathrm{d}\tau \doteqdot \frac{2\pi}{2n+1}\sum_{k=1}^{n}(1+\tau_k)f(\tau_k), \tag{5.3}$$

where

$$\tau_k = \cos\left(\frac{2k-1}{2n+1}\right)\pi, \quad k = 1,\ldots,n. \tag{5.4}$$

For the improper integral, the Gaussian quadrature formula is used. Thus

$$\int_0^\infty B(\eta)K_{12}(\xi,\eta)\,d\eta = \sum_{k=1}^{m} B(\eta_k)K_{12}(\xi,\eta_k)A_k, \tag{5.5}$$

where A_k are appropriate weights.

The solution of the integral equation is obtained by choosing the collocation points:

$$s_i = \cos\left(\frac{2i\pi}{2n+1}\right), \quad i = 1,\ldots,n \tag{5.6}$$

and solving the matrix system for $G^*(\tau_k)$, $\widetilde{B}(\xi_k)$ and $\widetilde{C}(\xi_k)$:

$$\sum_{k=1}^{n}[R(s_j,\tau_k) + S(s_j,\tau_k)]G^*(\tau_k) + \frac{2n+1}{2a}\sum_{k=1}^{m}\{\widetilde{B}(\xi_k)K_{12}(\xi_k,r_j) + \widetilde{C}(\xi_k)K_{13}(\xi_k,r_j)\}A_k = p(s_j)\frac{2n+1}{2a}, \quad j = 1,\ldots,n, \tag{5.7}$$

where $p(s_j)$ is the expression of the right-hand side of (5.1), and

$$G(\tau_k) = \frac{G^*(\tau_k)}{(1+\tau_k)[\frac{a}{2}(\tau_k+1)+1]} \tag{5.8}$$

and similar expression for (5.2).

We consider the kernel $S(r,t)$. It has a generalized Cauchy-type singularity, and should be also specified in order to improve the convergency of calculation. This is discussed in Atsumi and Shindo (1983b).

6. Numerical results and consideration

Numerical calculations have been carried out for $v = 0.3$. The values of normalized stress intensity factor $K/p_0\sqrt{a}$ versus a are shown in Figs. 2–4 for various values of c.

Fig. 2 shows the variation of $K/p_0\sqrt{a}$ with respect to a when $c = 4$. This figure shows that as a approaches to zero, the limiting value of $K/p_0\sqrt{a}$ is 2.295 which is in agreement with the value obtained by

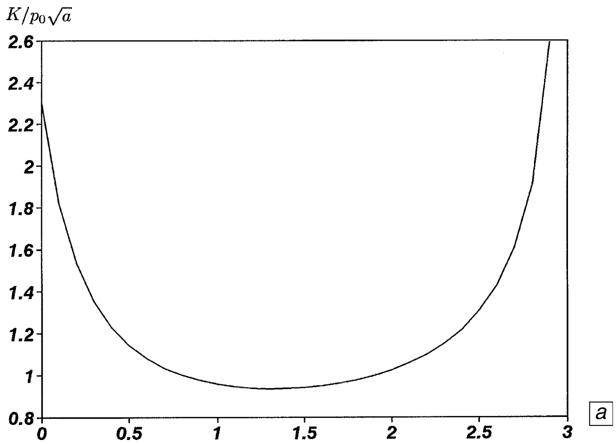

Fig. 2. Stress intensity factor for $c = 4$.

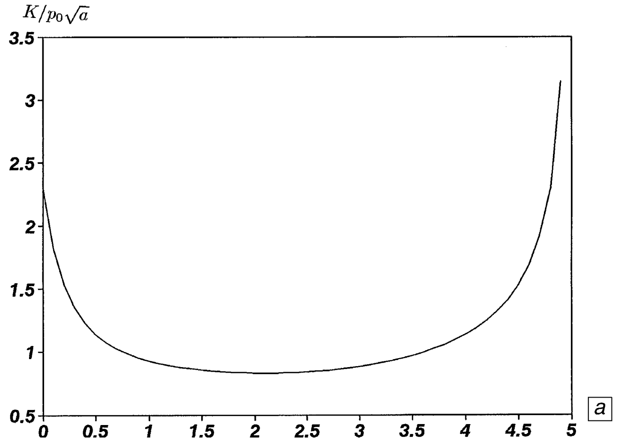

Fig. 3. Stress intensity factor for $c = 6$.

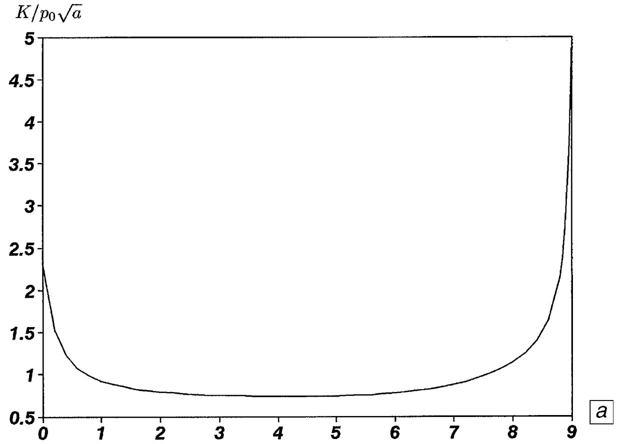

Fig. 4. Stress intensity factor for $c = 10$.

Atsumi and Shindo (1983b). This figure shows that as a increases, S.I.F. decreases until it achieves its minimum whose value is about 1 around the midpoint, and then begin to increase gradually until it increases very sharply. Figs. 3 and 4 deal with the cases when $c = 6$ and 10, respectively. We can see the trend is similar. In Fig. 4 we can notice that except both ends S.I.F. maintains relatively constant value.

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